A Hopf-bifurcation theorem for the critically dissipative quasi-geostrophic equation

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Abstract

This paper is devoted to the study of the dynamical behavior of the critically dissipative quasi-geostrophic equation in $\mathbf{T}^2$. We prove that this system possesses time-dependent periodic solutions, bifurcating from a smooth steady solution, i.e. a Hopf-Bifurcation theorem.

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1 Introduction and Main Results

We consider the two dimensional dissipative quasi-geostrophic equation

$$\partial_t\Theta + U\cdot\nabla\Theta + (-\triangle)^{\alpha}\Theta = f_\beta(x), \quad (x,t)\in\mathbf{T}^2\times\mathbf{R}^+, \tag{1.1}$$

where the unknown active scalar $\Theta(x,t)$ denotes the temperature on the boundary surface, $U(x,t)$ is the velocity vector and $f_\beta(x)$ is a given time independent smooth external force which is a $\mathbf{T}^2$-periodic function with zero mean and depends on some parameter $\beta>0$, $\mathbf{T}^2$ denotes a two dimensional torus. The velocity is coupled with the temperature via a stream function $\Psi(x,t)$:

$$\Theta = (-\triangle)^{\frac{1}{2}}\Psi = \Lambda\Psi, \tag{1.2}$$

and

$$U = \nabla^{\perp}\Psi = (\partial_{x_1}\Psi, -\partial_{x_2}\Psi) = (R_2\Theta, -R_1\Theta), \tag{1.3}$$

where R_i is the ith Riesz transform, Λ is the usual operator given by $\Lambda=(-\triangle)^{\frac{1}{2}}$ and defined via the Fourier transform as

$$\widehat{\Lambda^{\frac{s}{2}}f(y)} = |y|^s\hat{f}(y), \quad s\geq 0.$$

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Quasi-geostrophic equation describes the evolution of the temperature field on a surface that bounds the fluid. Equation (1.1) is special cases of the general quasi-geostrophic approximations [24] for atmospheric and oceanic fluid flow with small Rossby and Ekman numbers. It is also produced by Ekman layer pumping. From a mathematical point of view, when $\alpha = \frac{1}{2}$, ”dimensionally, the 2D quasi-geostrophic equation is the analogue of the 3D Navier-Stokes equations” (Constantin and Wu [7]), and the behaviour of solutions to (1.1) is similar to that of the 3D Navier-Stokes equations. Moreover, the much simpler QG equations in terms of possible formation of singularities. For this reason, $\alpha = \frac{1}{2}$ is considered the critical exponent, while $\alpha \in (\frac{1}{2}, 1]$ is the subcritical exponent, $\alpha \in (0, \frac{1}{2})$ is the supercritical exponent. When $\alpha = \frac{1}{2}$, (1.1) is the vorticity equation of the 2D Navier-Stokes equations. Recently the QG equation has been intensively investigated and much attention is carried to the problem of global existence and regularity. The results concerning regularity of solutions to the dissipative QG equation were given in the simpler (but non-physical) subcritical case where $\alpha > \frac{1}{2}$, see for example, Constantin and Wu [7]. They established the uniqueness of strong solutions and also showed that for $\theta_0 \in \mathbf{L}^1 \cap \mathbf{L}^2$,

$$\|\theta\|_{\mathbf{L}^2} \leq C(1+t)^{\frac{-1}{2\alpha}}, \;\; t \geq 0.$$

Córdoba and Córdoba [4] proved the decay of solutions

$$\|\theta\|_{\mathbf{L}^p} \leq C_1(1+C_2 t)^{\frac{-(p-1)}{2\alpha p}}, \;\; t \geq 0,$$

where C_1 and C_2 are explicit constants, the initial $\theta_0 \in \mathbf{L}^1 \cap \mathbf{L}^p$, $1 < p < \infty$. There are some other refining work about the existence and decay of solutions for dissipative QG (see [17, 18, 23, 27]).

For the critical case ($\alpha = \frac{1}{2}$), Constantin, Cordoba and Wu [6] proved the existence of a unique global solutions evolving from any initial data that are small in $\mathbf{L}^\infty$. Dong and Du [9] showed the global existence of solution for the critical dissipative QG equation with arbitrary $\mathbf{H}^1$ initial data and obtained the decay in time estimate for higher order homogeneous Sobolev norms of solutions. More recently, Caffarelli and Vasseur [2] used harmonic extension to establish regularity of the Leray-Hopf weak solution. Kiselev, Nazarov and Volberg [19] employed a certain non-local maximum principle for a suitably chosen modulus of continuity to prove the global well posedness of the critical dissipative QG equations with periodic $\mathcal{C}^\infty$ data. Both two papers remove the smallness assumption independently.

For the supercritical case ($\alpha \in (0, \frac{1}{2})$), the global existence of dissipative QG equation is still wide open. Silvestre [26] first showed that every weak solution of the dissipative SQG with slightly supercritical ($\alpha = \frac{1}{2} - \epsilon$, $\epsilon < \epsilon_0$) dissipative term becomes smooth after a certain time. Very recently, Dabkowski [8] improved previous result and showed that this eventual regularization property holds for arbitrarily small powers of the Laplacian.

Recently, Friedlander, Pavlović and Vicol [11] proved that a smooth steady state of the critical QG equation (1.1) is $(\mathbf{H}^1, \mathbf{L}^2)$ Lyapunov nonlinear unstable by assuming that the corresponding operator has spectrum in the unstable region. Meanwhile, they obtained that the forced QG equation has a unique global smooth solution:

Theorem 1.1. *(Friedlander, Pavlović and vicol [11]) Assume that Θ_0 and f in $\mathcal{C}^\infty$ are $\boldsymbol{T}^2$-periodic functions with zero mean. Then there exists a unique global in time smooth solution*

of

$$\partial_t\Theta + U\cdot\nabla\Theta + (-\triangle)^{\alpha}\Theta = f(x),$$
$$U = \nabla^{\perp}\Psi = (R_2\Theta, -R_1\Theta),$$
$$\Theta(0) = \Theta_0.$$

Moreover for all $t \geq 0$, we have

$$\|\nabla\Theta\|_{\boldsymbol{L}^\infty} \leq C_0,$$

where $C_0 = C_0(\|\Theta_0\|_{\boldsymbol{L}^\infty}, \|\nabla\Theta_0\|_{\boldsymbol{L}^\infty}, \|f\|_{\boldsymbol{L}^\infty}, \|\nabla f\|_{\boldsymbol{L}^\infty})$ is a positive constant.

For the subcritical dissipative QG equation, above nonlinear instability result can be directly proved by the arguments in [10].

From Theorem 1.1, assume that Θ_β (steady state solution) and $f_\beta(x)$ in $\mathcal{C}^\infty$ are $\mathbf{T}^2$-periodic functions with zero mean, we know that equation (1.1) has a unique smooth solution Θ such that $\|\nabla\Theta\|_{\mathbf{L}^\infty} < \infty$ for every fixed $\beta > 0$. In the present paper, we show that the solution Θ is a periodic solution. More precisely, we study the existence of time periodic solution which bifurcating from a smooth steady state for the smooth parameter external forced critical QG equations. This can be regarded as a Hopf-Bifurcation theorem for the critical QG equation. There are two difficulties in dealing with our problem. Firstly, the appearance of a continuous spectrum up to the imaginary axis of the operator **F** (see (1.7)). Two complex conjugate eigenvalues with a nonvanishing imaginary part cross the imaginary axis at $\beta = \beta_c$. This leads to the trivial solution becomes unstable. Secondly, due to the relation between the temperature and the stream function, the Lyapunov-Schmidt decomposition can not direct apply to find time periodic solution for the critical QG equation.

From a general point of view, the occurrence of time dependent periodic flows is the inherent phenomenon to exist in the viscous incompressible fluid motions. In particular, the transition from a steady-state flow to time dependent periodic flows or Hopf bifurcation may be considered to be the primary stage in understanding the nature of turbulence (see [20, 21]). The study of the bifurcation of stationary solutions into time periodic solutions (i.e. Hopf-bifurcation) of incompressible Navier-Stokes equation started from the work of Sattinger [25], Iudovich [16] and Iooss [13] in 1971. After that there is much work concerning Hopf-Bifurcation problem of incompressible Navier-Stokes equations, see [1, 3, 14, 15], etc. We mention that Crandall and Rabinowitz [5] gave an abstract infinite-dimensional version of Hopf bifurcation theorem which has became a basic tool in applications. But we can not directly use the method of dealing with Navier-Stokes equation to the critical QG equation because the relation between the temperature and the stream function is not equivalent to the relation between the vorticity and the stream function in the 3D Navier-Stokes equations.

In the present paper, we obtain the following Hopf bifurcation result on the critical dissipative QG equation.

Theorem 1.2. *Assume that $c > 0$, the cubic coefficient $\mu \neq 0$ in (3.22) and **(H1)** $-$ **(H3)** hold. Then (1.1) has a one-dimensional family of small time-periodic smooth solutions, i.e., there exists $\epsilon_0 > 0$ such that*

$$\Theta(x,t) = \Theta(x, t + 2\pi/\omega),$$

for all $\epsilon \in]0, \epsilon_0[$ *and* $\beta = \beta_c + \epsilon^2$*. Moreover, it holds*

$$\omega = \omega_0 + o(\epsilon)$$

and

$$\|\nabla\Theta\|_{\boldsymbol{L}^\infty} \leq C_0,$$

where $C_0 = C_0(\|\theta_\beta\|_{\boldsymbol{L}^\infty}, \|\nabla\theta_\beta\|_{\boldsymbol{L}^\infty}, \|f_\beta\|_{\boldsymbol{L}^\infty}, \|\nabla f_\beta\|_{\boldsymbol{L}^\infty})$ *is a positive constant.*

Throughout this paper, we assume that there exist $\beta_c \in \mathbf{R}^+$ and $\delta_0 > 0$ such that

(H1) $\lambda = 0$ is not an eigenvalue of $\mathbf{F}$ for any value of $\beta \in [\beta_c - \delta_0, \beta + \delta_0]$.

(H2) For $\beta = \beta_c > 0$ the operator $\mathbf{F}$ has two eigenvalues $\lambda_0^{\pm}(\beta)$ satisfying

$$\lambda_0^{\pm}(\beta_c) = \pm i\omega_0 \neq 0 \;\; and \;\; \frac{d}{d\beta} Re(\lambda_0^{\pm}(\beta))|_{\beta=\beta_c} > 0.$$

(H3) For any $\beta \in [\beta_c - \delta_0, \beta_c + \delta_0]$, all other eigenvalues of $\mathbf{F}$ are strictly bounded away from the imaginary axis in the left half plane.

We also assume that the external force f_β depends smoothly on some parameters β and such that there exists a smooth stationary solution θ_β with velocity q_β (We do not assume that q_β is smooth)

$$q_\beta \cdot \nabla\theta_\beta + \Lambda\theta_\beta = f_\beta(x), \tag{1.4}$$

$$q_\beta = (R_2\theta_\beta, -R_1\theta_\beta). \tag{1.5}$$

Furthermore, we assume that

$$q_\beta(x) = q_c + q_0(x) \tag{1.6}$$

with

$$\lim_{|x|\longrightarrow\infty} \theta_\beta(x) = 0, \quad \lim_{|x|\longrightarrow\infty} q_0(x) = 0,$$

where $q_c = (c, 0)^T$. We linearize (1.1) about the steady state (θ_β, q_β) by writing

$$\Theta(x,t) = \theta_\beta + \theta(x,t), \;\; U(x,t) = q_\beta + q(x,t).$$

Thus, we obtain an equation that governs the perturbation θ:

$$\partial_t\theta = \mathbf{F}\theta + N(\theta),$$

where the linear operator $\mathbf{F}$ is defined by

$$\mathbf{F}\theta = -q_c \cdot \nabla\theta - q_0 \cdot \nabla\theta - \Lambda\theta, \tag{1.7}$$

the velocity is coupled with the temperature via

$$q = (R_2\theta, -R_1\theta), \tag{1.8}$$

and

$$N(\theta) = -q \cdot \nabla\theta - q \cdot \nabla\theta_\beta = -q \cdot \nabla\Theta. \tag{1.9}$$

Using Fourier transform and Riesz Theorem, we can know that the operator $\mathbf{F}$ is a relatively compact perturbation of $\mathbf{F}_0 = -q_c \cdot \nabla\theta - \Lambda\theta$. The spectra of $\mathbf{F}_0 = -q_c \cdot \nabla\theta - \Lambda\theta$ and $\mathbf{F}$ only differ by isolated eigenvalues of finite multiplicity (see [12]). Hence the essential spectrum of $\mathbf{F}$ equals the essential spectrum of $\mathbf{F}_0$, i.e.

$$essspec(\widehat{\mathbf{F}_0}) = \{\lambda \in \mathcal{C} : \lambda = -|y| + icy_1, \;\; y \in \mathbf{T}^2\}.$$

Here we denote β as the bifurcation parameter. To overcome the essential spectrum of operator $\mathbf{F}_0$ up to the imaginary axis, classical Hopf-bifurcation condition **(H2)** is needed (see [25]).

This paper is organized as follows. In section 2, we introduce some notations and preliminaries. The proof of the main result Theorem 1.2 is carried out in section 3.

2 Notations and Preliminaries

We denote $\mathbf{H}^m = \mathbf{H}^m(\mathbf{T}^2, \mathbf{C})$ by the usual Sobolev space. The Fourier thransform is an isomorphism between the spaces $\mathbf{H}^m$ and $\mathbf{L}^2_m$, the latter is defined via the norm $\|\theta\|_{\mathbf{L}^2_m} = \|\rho^m\theta\|_{\mathbf{L}^2}$ with the weighted function $\rho(x) = \sqrt{1+|x|^2}$ or $\rho(x,\epsilon) = |x|\sqrt{1+\epsilon|x|}$. For fixed $m \in \mathbf{N}$ such that $m > 1$, we will prove our main result for $\theta = (\theta_n)_{n\in\mathbf{Z}}$ in the space

$$\mathcal{X} = \{\theta = (\theta_n)_{n\in\mathbf{Z}} : \|\theta\|_{\mathcal{X}} < \infty\}$$

with the norm defined by

$$\|\theta\|_{\mathcal{X}} = \sum_{n\in\mathbf{Z}} \|\theta_n\|_{\mathbf{H}^m}.$$

We consider $\omega > 0$ close to ω_0 defined in **(H2)**, and we find $2\pi/\omega$-time periodic solutions of

$$\partial_t\theta = \mathbf{F}\theta + N(\theta), \tag{2.1}$$

which is introduced in section 1. To seek the time periodic solution of the form $\theta = \theta(x, t/2\pi)$, we consider the equation

$$\omega\partial_t\theta = \mathbf{F}\theta + N(\theta).$$

Due to the periodicity in time we can make the ansatz

$$\theta(x,t) = \sum_{n\in\mathbf{Z}} \theta_n(x)e^{int}.$$

Then we obtain the system

$$(in\omega - \mathbf{F})\theta_n = N_n(\theta), \tag{2.2}$$

where we set

$$N(\theta)(x,t) = \sum_{n\in\mathbf{Z}} N_n(\theta)(x)e^{int}. \tag{2.3}$$

Note that we are interested in real valued solutions only. We will always suppose that $\overline{\theta_n} = \theta_{-n}$ for $n \in \mathbf{Z}$. These series are uniformly convergent on $\mathbf{T}^2 \times [0, 2\pi]$ in the chosen spaces. More precisely, we have the following result:

Lemma 2.1. *A linear operator* $\boldsymbol{J} : \mathcal{X} \longrightarrow \boldsymbol{C}_b^0(\boldsymbol{T}^2 \times [0,\pi], \boldsymbol{C}^2)$ *is defined by*

$$(\boldsymbol{J}\theta)(x,t) = \tilde{\theta}(x,t) := \sum_{n\in\mathbf{Z}} \theta_n(x)e^{int}, \ \ \theta = (\theta_n)_{n\in\mathbf{Z}} \in \mathcal{X}.$$

Then $\boldsymbol{J}$ *is bounded.*

The counterpart to multiplication $\theta^1\theta^2$ in physical space is given by the convolution $(\sum_{k\in\mathbf{Z}} \theta^1_{n-k}\theta^2_k)_{n\in\mathbf{Z}}$, since

$$\theta^1\theta^2 = \sum_{l\in\mathbf{Z}} \theta^1_l(x)e^{ilt} \sum_{j\in\mathbf{Z}} \theta^2_j(x)e^{ijt} = \sum_{n\in\mathbf{Z}} \left(\sum_{k\in\mathbf{Z}} \theta^1_{n-k}(x)\theta^2_k(x) \right) e^{int}.$$

Lemma 2.2. *For* $\theta^1 = (\theta^1_n)_{n\in\mathbf{Z}}, \ \theta^2 = (\theta^2_n)_{n\in\mathbf{Z}} \in \mathcal{X}$*, the convolution* $\theta^1 * \theta^2 \in \mathcal{X}$ *is defined by*

$$(\theta^1 * \theta^2)_n = \sum_{k\in\mathbf{Z}} \theta^1_{n-k}\theta^2_k, \ \ n \in \mathbf{Z}.$$

Then there exists $C > 0$ *such that*

$$\|\theta^1 * \theta^2\|_{\mathcal{X}} \le C\|\theta^1\|_{\mathcal{X}}\|\theta^2\|_{\mathcal{X}}.$$

Lemma 2.3. *Let the linear operator* $\boldsymbol{F} : \mathcal{X} \longrightarrow \mathcal{X}$ *be defined component-wise as* $(\boldsymbol{F}\theta)_n = \boldsymbol{F}_n\theta_n$ *for* $\theta = (\theta_n)_{n\in\boldsymbol{Z}}$*. Then*

$$\|\boldsymbol{F}\theta\|_{\mathcal{X}} = (\|\boldsymbol{F}_0\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} + \sup_{n\in\boldsymbol{Z}\setminus\{0\}} \|\boldsymbol{F}_n\|_{\boldsymbol{H}^m_s \longrightarrow \boldsymbol{H}^m_s})\|\theta\|_{\mathcal{X}}.$$

The proof of the above three Lemmas are rather standard, so we omit it.

3 Proof of Theorem 1.2

In this section, we will give the detail of proof of Theorem 1.2. By **(H2)** and **(H3)**, we know that the operator $\mathbf{F}$ has two eigenvalues $\lambda_0^{\pm}(\beta)$ and all other eigenvalues of $\mathbf{F}$ are strictly bounded away from the imaginary axis in the left half plane. Thus we construct an $\mathbf{F}$-invariant projections $\mathbf{P}_{\pm 1,c}$ by

$$P_{1,c}\theta = (\psi^{+,*},\theta)_{\mathbf{L}^2}\psi^+, \;\; P_{-1,c}\theta = (\psi^{-,*},\theta)_{\mathbf{L}^2}\psi^-, \tag{3.1}$$

where $\psi^{\pm}$ denote the associated normalized eigenfunctions, $\psi^{\pm 1,*}$ denotes the associated normalized eigenfunctions of the adjoint operator $\mathbf{F}^*$. The bounded "stable" part of the projection is $\mathbf{P}_{\pm 1,s} = I - \mathbf{P}_{\pm 1,c}$ and we also know that $\mathbf{P}_{\pm,c}\mathbf{F} = \mathbf{F}\mathbf{P}_{\pm,c}$ and $\mathbf{P}_{\pm,s}\mathbf{F} = \mathbf{F}\mathbf{P}_{\pm,s}$. Thus we can split $\theta_{\pm 1}$ as $\theta_1 = \theta_{1,c} + \theta_{1,s}$ and $\theta_{-1} = \theta_{-1,c} + \theta_{-1,s}$ with $\theta_{\pm 1,c} = \mathbf{P}_{\pm 1,c}\theta_1$ and $\theta_{\pm 1,s} = \mathbf{P}_{\pm 1,s}\theta_1$. Using the above decompositions to (2.2), we have

$$(in\omega - \mathbf{F})\theta_n = -N_n(\theta), \;\; n = \pm 2, \pm 3, \ldots, \tag{3.2}$$
$$\mathbf{F}\theta_0 = -N_0(\theta), \;\; n = 0, \tag{3.3}$$
$$(\pm i\omega - \mathbf{F})\theta_{\pm 1,s} = \mathbf{P}_{\pm 1,s}N_{\pm 1}(\theta), \tag{3.4}$$
$$(\pm i\omega - \mathbf{F})\theta_{\pm 1,c} = \mathbf{P}_{\pm 1,c}N_{\pm 1}(\theta). \tag{3.5}$$

The organization of the proof is that we first solve the equation (3.3). Then using the fixed point theorem to solve equations (3.2) and (3.4) which is nontrivial due to the nonlinear term $N_n(\theta)$. Thanks to the explicit representation (see [4]) and Fourier transform property, we can estimate the nonlinear term and solve equations (3.2) and (3.4) by s small modification of equations (3.2) and (3.4). At last, we employ the implicit function theorem to solve equation (3.5). The process of solving equation (3.5) is inspired by the classical Hopf-Bifurcation result [22].

Now we first solve the equation (3.3). Denote $\mathbf{F}_0 = -q_c \cdot \nabla\theta - \Lambda\theta$, then $\mathbf{F} = \mathbf{F}_0 - q_0 \cdot \nabla\theta$. Although the linear operator $\mathbf{F}_0$ has the essential spectrum up to the imaginary axis, it can be be inverted in the following sense.

Lemma 3.1. *For $j = 1, 2$ and $f \in \boldsymbol{H}^m \cap \boldsymbol{L}^1$, the equation*

$$\boldsymbol{F}_0\theta = \partial_j f$$

has a unique solution $\theta = \boldsymbol{F}_0^{-1}\partial_j f \in \boldsymbol{H}^m$. Moreover,

$$\|\theta\|_{\boldsymbol{H}^m} \le C\|f\|_{\boldsymbol{H}^m}.$$

Proof. Define a smooth cut-off function χ taking its value in $[0,1]$ as

$$\chi(y) := \begin{cases} 1, & |y| \le 1, \\ 0, & |y| \ge 2. \end{cases}$$

We denote $\hat{f}_1 = \hat{f}\chi$, $\hat{f}_2 = \hat{f}(1-\chi)$ with $\hat{f} = \hat{f}_1 + \hat{f}_2$, and

$$\hat{\theta}_1(y) = \frac{iy_j\hat{f}_1}{-|y| + icy_1} \;\; and \;\; \hat{\theta}_2(y) = \frac{iy_j\hat{f}_2}{-|y| + icy_1}.$$

Then $\theta = \theta_1 + \theta_2$. Moreover,

$$\begin{aligned}
\|\theta_1\|^2_{\mathbf{H}^m} &= \|\hat{\theta}_1\|^2_{\mathbf{L}^2_m} = \int_{\mathbf{R}^2} \frac{|y_j|^2 |\hat{f}\chi(y)|^2}{|-|y|+icy_1|^2} \rho^{2m}(y)dy \\
&\leq C\|f\|^2_{\mathbf{L}^1} \int_{|y|\leq 2} \frac{|y_j|^2}{r^2+c^2y_1^2} dy \\
&\leq C\|f\|^2_{\mathbf{L}^1},
\end{aligned}$$

and

$$\begin{aligned}
\|\theta_2\|^2_{\mathbf{H}^m} &= \|\hat{\theta}_2\|^2_{\mathbf{L}^2_m} = \int_{\mathbf{R}^2} \frac{|y_j|^2 |\hat{f}(1-\chi(y))|^2}{|-|y|+icy_1|^2} \rho^{2m}(y)dy \\
&\leq C\int_{\mathbf{R}^2} |\hat{f}(y)|^2 \rho^{2m}(y)dy \\
&\leq C\|f\|^2_{\mathbf{H}^m}.
\end{aligned}$$

This completes the proof. □

This Lemma tells us that $\widehat{\mathbf{F}_0} iy_j \cdot$ is a bounded compact operator from $\mathbf{L}^2_m$ to itself. Furthermore, the spectra of $\widehat{\mathbf{F}}$ and $\widehat{\mathbf{F}_0}$ only differ by isolated eigenvalues of finite multiplicity (see the book of Henry [12] p.136).

The following lemma gives the solvability of the equation (3.3).

Lemma 3.2. *Assume that* ***(H1)****-****(H3)*** *hold. Then the equation (3.3) has a unique solution*

$$\theta_0 = -\boldsymbol{F}^{-1} N_0(\theta). \tag{3.6}$$

Moreover,

$$\|\theta_0\|_{\boldsymbol{H}^m} \leq C\|y_j^{-1}\widehat{N_0(\theta)}\|_{\mathbf{L}^2_m}.$$

Proof. Since the operator $\widehat{\mathbf{F}_0}^{-1}\widehat{\mathbf{F}_1} : \mathbf{L}^2_m \longrightarrow \mathbf{L}^2_m$ is compact, the operator $I + \widehat{\mathbf{F}_0}^{-1}\widehat{\mathbf{F}_1}$ is Fredholm with index 0. If $(I + \widehat{\mathbf{F}_0}^{-1}\widehat{\mathbf{F}_1})\hat{\theta} = 0$ has a nontrivial solution, then $\widehat{\mathbf{F}}\hat{\theta} = \widehat{\mathbf{F}_0}(I + \widehat{\mathbf{F}_0}^{-1}\widehat{\mathbf{F}_1})\hat{\theta} = 0$ would also have a nontrivial solution. This would contradict (**H1**). Hence the Fredholm property implies that the existence of $(I + \widehat{\mathbf{F}_0}^{-1}\widehat{\mathbf{F}_1})^{-1} : \mathbf{L}^2_m \longrightarrow \mathbf{L}^2_m$. Note that $\widehat{\mathbf{F}} = \widehat{\mathbf{F}_0} + \widehat{\mathbf{F}_1}$ with $\widehat{\mathbf{F}_1} = iy\hat{q}_0 * \cdot$. Then we have

$$\widehat{\mathbf{F}_0}(I + \widehat{\mathbf{F}_0}^{-1}\widehat{\mathbf{F}_1})\hat{\theta} = iy_j\hat{f},$$

where $\hat{f} = -iy_j^{-1}\widehat{N_0(\theta)}$.

Thus, by Lemma 3.1, we obtain

$$\begin{aligned}
\|\theta\|_{\mathbf{H}^m} = \|\hat{\theta}\|_{\mathbf{L}^2_m} &\leq \|(I + \widehat{\mathbf{F}_0}^{-1}\widehat{\mathbf{F}_1})^{-1}\|_{\mathbf{L}^2_m \longrightarrow \mathbf{L}^2_m} \|\widehat{\mathbf{F}_0}^{-1} iy_j\hat{f}\|_{\mathbf{L}^2_m} \\
&\leq C\|\hat{f}\|_{\mathbf{L}^2_m}.
\end{aligned}$$

This completes the proof. □

To solve equations (3.2) and (3.4), we consider the following equations

$$(in\omega - \mathbf{F})\Lambda^{-1}\theta_n = -\Lambda^{-1}N_n(\theta), \quad n = \pm 2, \pm 3, \ldots, \tag{3.7}$$
$$(\pm i\omega - \mathbf{F})\Lambda^{-1}\theta_{\pm 1,s} = \mathbf{P}_{\pm 1,s}\Lambda^{-1}N_{\pm 1}(\theta). \tag{3.8}$$

The following result gives the bounds on nonlinear terms.

Lemma 3.3. *Define $N : \mathcal{X} \longrightarrow \mathcal{X}$ by $N(\theta)_n = N_n(\boldsymbol{J}\theta)$ for $\theta \in \mathcal{X}$. Then there exists $C > 0$ such that*

$$\|\Lambda^{-1}N(\theta)\|_{\mathcal{X}} \leq C\|\theta\|_{\mathcal{X}}^2, \tag{3.9}$$

for $\theta \in \mathcal{X}$ with $\|\theta\|_{\mathcal{X}} \leq 1$. Moreover, there exists $C > 0$ such that

$$\|\Lambda^{-1}N(\theta^1) - \Lambda^{-1}N(\theta^2)\|_{\mathcal{X}} \leq C(\|\theta^1\|_{\mathcal{X}} + \|\theta^2\|_{\mathcal{X}})\|\theta^1 - \theta^2\|_{\mathcal{X}}, \tag{3.10}$$

for $\theta^1,\ \theta^2 \in \mathcal{X}$ with $\|\theta^1\|_{\mathcal{X}} \leq 1$ and $\|\theta^2\|_{\mathcal{X}} \leq 1$.

Proof. In order to estimate the factor $\|\Lambda^{-1}N(\theta)\|_{\mathcal{X}}$, we recall the explicit representation (see [4]) of the nonlinear term

$$\begin{aligned}
\Lambda^{-1}N(\theta) &= \Lambda^{-1}(R(\theta)\cdot\nabla\theta) = C_n(R_1(\theta R_2(\theta)) - R_2(\theta R_1(\theta))) \\
&= C_n\left(R_1(\sum_{n\in\mathbf{Z}}\theta_n e^{int}R_2(\sum_{n\in\mathbf{Z}}\theta_n e^{int})) - R_2(\sum_{n\in\mathbf{Z}}\theta_n e^{int}R_1(\sum_{n\in\mathbf{Z}}\theta_n e^{int}))\right) \\
&= C_n\sum_{n\in\mathbf{Z}}\left(\sum_{k\in\mathbf{Z}}(R_1(\theta_{n-k}(x)R_2\theta_k(x)) - R_2(\theta_{n-k}(x)R_1\theta_k(x)))\right)e^{int}
\end{aligned} \tag{3.11}$$

for some dimensional constant C_n.

By (3.11) and Fourier transform property, for $m \geq 4$, we derive

$$\begin{aligned}
\|\Lambda^{-1}N_n(\theta)\|_{\mathbf{H}^m} &= \|\sum_{k\in\mathbf{Z}}(R_1(\theta_{n-k}(x)R_2\theta_k(x)) - R_2(\theta_{n-k}(x)R_1\theta_k(x))\|_{\mathbf{H}^m} \\
&\leq C\|\sum_{k\in\mathbf{Z}}(\frac{y_1y_2}{|y|^2}\widehat{\theta_{n-k}(y)} * \widehat{\theta_k(y)})\|_{\mathbf{L}^2_m} \\
&\leq C\|\sum_{k\in\mathbf{Z}}(|y|^{\frac{m}{2}}(1+\epsilon|y|)^{\frac{m}{2}}\frac{y_1y_2}{|y|^2}\widehat{\theta_{n-k}(y)} * \widehat{\theta_n(y)})\|_{\mathbf{L}^2} \\
&\leq C\|\sum_{k\in\mathbf{Z}}(y_1y_2|y|^{\frac{m}{2}-2}(1+\epsilon|y|)^{\frac{m}{2}}\widehat{\theta_{n-k}(y)} * \widehat{\theta_n(y)})\|_{\mathbf{L}^2} \\
&\leq C\|\sum_{k\in\mathbf{Z}}(|y|^{\frac{m}{2}}(1+\epsilon|y|)^{\frac{m}{2}}\widehat{\theta_{n-k}(y)} * \widehat{\theta_n(y)})\|_{\mathbf{L}^2} \\
&\leq C\|\sum_{k\in\mathbf{Z}}\theta_{k-n}\theta_k\|_{\mathbf{H}^m}^2,
\end{aligned} \tag{3.12}$$

where we take the weighted norm as $\rho(x,\epsilon) = |x|^{\frac{1}{2}}\sqrt{1+\epsilon|x|}$.

Note that a contribution $\tilde{\theta}^k = (\mathbf{J}\theta)^k$ to $N(\theta) = N(\mathbf{J}\theta)$ leads to the nth coefficient $(\theta * \ldots * \theta)_n$ with k-fold convolution. Hence, by Lemma 2.2 and (3.12), we conclude

$$\begin{aligned}\|\Lambda^{-1}N(\theta)\|_{\mathcal{X}} &= \sum_{n\in\mathbf{Z}} \|\Lambda^{-1}N_n(\mathbf{J}\theta)\|_{\mathbf{H}^m} \\ &\leq C\|\theta\|_{\mathcal{X}}^2.\end{aligned}$$

By the similar proof, we can get (3.10). $\square$

Lemma 3.4. *There exits $C > 0$ such that for ω close enough to ω_0 the following estimates hold:*

$$\begin{aligned}&\|(in\omega - \boldsymbol{F}_0)^{-1}\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \leq C, \ \ n \neq 0, \\ &\|(in\omega - \boldsymbol{F})^{-1}\Lambda\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \leq C, \ \ n \neq 0, \\ &\|(i\omega - \boldsymbol{F}_0)^{-1}\boldsymbol{P}_{\pm1,s}\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \leq C, \\ &\|(i\omega - \boldsymbol{F})^{-1}\Lambda\boldsymbol{P}_{\pm1,s}\|_{\boldsymbol{H}^m \longrightarrow \boldsymbol{H}^m} \leq C.\end{aligned}$$

Proof. We observe that the solution θ of the equation $(in\omega - \mathbf{F}_0)\theta = f$ is given by

$$\widehat{\theta}(y) = (in\omega + |y| - icy_1)^{-1}\widehat{f}(y), \ \ y \in \mathcal{R}^2.$$

Note that for $\delta = \frac{\omega^2}{\omega^2+4c^2}$,

$$|in\omega + |y| - icy_1|^2 = |y|^2 + (cy_1 + n\omega)^2 \geq \frac{\omega^2}{4c^2}\chi_{|y|\leq\frac{\omega}{2c}} + \delta^2(1+|y|^2)\chi_{|y|\geq\frac{\omega}{2c}}.$$

It follows for $f \in \mathbf{H}^m$ that $\widehat{\theta} \in \mathbf{L}^2_{m+1}$, thus $\theta \in \mathbf{H}^{m+1}$.

Let $\hat{f} \in \mathbf{L}^2_{m+1} \subset \mathbf{L}^2_m$ and $\widehat{\bar{\theta}} = \rho(y,\epsilon)\hat{\theta}$ with $\rho(x,\epsilon) = \sqrt{1+\epsilon|x|^2}$. Note that θ is a solution of the equation $(in\omega - \mathbf{F}_0)\theta = f$. By a direct computation, we have

$$(in\omega + |y| - icy_1)\widehat{\bar{\theta}} + \epsilon L(y,\epsilon)\widehat{\bar{\theta}} = \hat{g},$$

where

$$\epsilon L(y,\epsilon) = (in\omega + |y| - icy_1)(1 - \rho^{-1}(y,\epsilon)),$$

and $\hat{g} = \rho(y,\epsilon)\hat{f}$. Here we have used the fact that $\mathbf{F}_0$ is elliptic of order 1. Hence it derives from the form of $\rho(y,\epsilon) = \sqrt{1+\epsilon|y|^2}$ that

$$L(y,\epsilon) \sim \frac{\epsilon|y|^3}{1+\epsilon|y|^2 + \sqrt{1+\epsilon|y|^2}}.$$

Using a Neumann series, it derives from the boundness of the operator $L : \mathbf{L}^2_{m+1} \longrightarrow \mathbf{L}^2_m$ that

$$(in\omega - \widehat{\mathbf{F}_0}) + \epsilon L : \mathbf{L}^2_{m+1} \longrightarrow \mathbf{L}^2_m$$

is invertible with a bounded inverse, for sufficiently small $\epsilon > 0$. This implies that $\bar{\theta} \in \mathbf{L}^2_{m+1}$, i.e., $\theta \in \mathbf{H}^{m+2}$. Moreover, we have

$$\|\theta\|_{\mathbf{H}^{m+2}} = \|\hat{\theta}\|_{\mathbf{L}^2_{m+2}} = \|\widehat{\bar{\theta}}\|_{\mathbf{L}^2_{m+1}} \leq C\|\hat{g}\|_{\mathbf{L}^2_m} = C\|f\|_{\mathbf{H}^{m+1}}.$$

The above result shows that $(in\omega - \mathbf{F}_0)^{-1} : \mathbf{H}^m \longrightarrow \mathbf{H}^{m+1}$ is bounded. But we only need this operator to be bounded in sense of $\mathbf{H}^m \longrightarrow \mathbf{H}^m$. This implies that the spectrum of $\mathbf{F}_0$ in $\mathbf{H}^m$ well separated from $in\omega$ for $n \neq 0$ and $\epsilon > 0$ sufficiently small. In a similar manner to prove the first inequality, the rest three inequalities can be obtained, so we omit it. This completes the proof. ☐

In what follows, we solve equations (3.7)-(3.8) in terms of the $\theta_{\pm 1,c}$. Rewrite equations (3.2)-(3.4) as

$$\theta_n = -(in\omega - \mathbf{F})^{-1}\Lambda(\Lambda^{-1}N_n(\theta)), \ \ n = \pm 2, \pm 3, \ldots, \tag{3.13}$$

$$\theta_{\pm 1,s} = (\pm i\omega - \mathbf{F})^{-1}\Lambda \mathbf{P}_{\pm 1,s}\Lambda^{-1}N_{\pm 1}(\theta), \tag{3.14}$$

with respect to the θ_n for $n \neq \pm 1$, and $\theta_{\pm 1,s}$.

Lemma 3.5. *Assume that there exist $\sigma_1, \sigma_2 > 0$ such that for all $\omega > 0$ with $|\omega - \omega_0| \leq \sigma_1$ and all $\theta_{\pm 1,c} \in \boldsymbol{H}^m$ with $\|\theta_{\pm 1,c}\|_{\boldsymbol{H}^m} \leq \sigma_2$. Then equations (3.13)-(3.14) have unique solutions $\tilde{\theta} = \Phi(\theta_c) \in \mathcal{X}$, where $\theta_c = (\theta_{-1,c}, \theta_{1,c})$ and $\tilde{\theta} = (\ldots, \theta_{-2}, \theta_{-1,c} + \theta_{-1,s}, \theta_0, \theta_{1,c} + \theta_{1,s}, \theta_2, \ldots)$. Moreover, there exits $C > 0$ such that*

$$\Phi(0) = 0, \ \ \|\tilde{\theta} - \theta_c\|_{\mathcal{X}} \leq C(\|\theta_{-1,c}\|^2_{\boldsymbol{H}^m} + \|\theta_{1,c}\|^2_{\boldsymbol{H}^m}), \tag{3.15}$$

and

$$\|\tilde{\theta}\|_{\mathcal{X}} \leq C(\|\theta_{-1,c}\|^2_{\boldsymbol{H}^m} + \|\theta_{1,c}\|^2_{\boldsymbol{H}^m}). \tag{3.16}$$

with $\tilde{\theta} - \theta_c := (\ldots, , 0, \theta_{-1,c}, 0, \theta_{1,c}, 0, \ldots)$,

Proof. For fixed $\omega > 0$ close to ω_0 and given $\theta_{\pm 1,c} \in \mathbf{H}^m$ with $\|\theta_{\pm 1,c}\|_{\mathbf{H}^m} \leq \sigma_2$, define the operator

$$\begin{aligned} \Gamma : \tilde{\theta}^* \ \ &\longmapsto \ \ \tilde{\theta} = \tilde{\theta}^* + (\ldots, 0, \theta_{-1,c}, 0, \theta_{1,c}, 0, \ldots) \\ &\longmapsto \ \ \theta \longmapsto \tilde{\theta}^{**} = right\ hand\ side\ of\ (3.13) - (3.14), \end{aligned}$$

where $\theta = \mathbf{J}\tilde{\theta}$ is defined in Lemma 3.1 and

$$\begin{aligned} &\tilde{\theta}^* = (\ldots, \theta_{-2}, \theta_{-1,s}, \theta_0, \theta_{1,s}, \theta_2, \ldots), \\ &\tilde{\theta} = \tilde{\theta}^* + \theta_c = \tilde{\theta}^* + (\ldots, 0, \theta_{-1,c}, 0, \theta_{1,c}, 0, \ldots). \end{aligned}$$

Now we prove the operator Γ is a self-map of a sufficiently small ball in $\mathcal{X}$. Using Lemmas 3.2-3.4, we have

$$\begin{aligned} \|\tilde{\theta}^{**}\|_{\mathcal{X}} \ &\leq \ C\sup\{\|(in\omega - \mathbf{F})^{-1}\Lambda\|_{\mathbf{H}^m \longrightarrow \mathbf{H}^m}, \|(\pm i\omega - \mathbf{F})^{-1}\Lambda \mathbf{P}_{\pm 1,s}\|_{\mathbf{H}^m \longrightarrow \mathbf{H}^m} \\ &\quad\ : n \in \mathbf{Z} \setminus \{-1, 1\}\} \times \|(\Lambda^{-1}N_n(\theta))_{n \in \mathbf{Z}}\|_{\mathcal{X}} \\ &\leq \ C\|\Lambda^{-1}N(\theta)\|_{\mathcal{X}} \leq C(\|\tilde{\theta}^*\|^2_{\mathcal{X}} + \|\theta_{-1,c}\|^2_{\mathbf{H}^m} + \|\theta_{1,c}\|^2_{\mathbf{H}^m}) \\ &\leq \ C(\|\tilde{\theta}^*\|^2_{\mathcal{X}} + \sigma_2^2), \end{aligned} \tag{3.17}$$

which implies that for sufficiently small $\sigma_2 > 0$, Γ maps the $\|\cdot\|_{\mathcal{X}}$ ball of radius $r = 1$ to itself. Then using the similar proof process of (3.10), we conclude that Γ is a contraction. Hence,

we obtain a unique fixed point $\tilde{\theta}^* \in \mathcal{X}$ of Γ, which means that equations (3.13)-(3.14) have solutions of $\tilde{\theta} = \tilde{\theta}^* + \theta_c$. Moreover, if $\theta_{\pm 1,c} = 0$, then $\Phi(0) = 0$. Next we prove the second inequality in (3.15). Note that

$$\tilde{\theta}^* = \Gamma(\tilde{\theta}^*) = \tilde{\theta}^{**},$$

which together with (3.17) implies

$$\|\tilde{\theta} - \theta_c\|_{\mathcal{X}} = \|\tilde{\theta}^*\|_{\mathcal{X}} = \|\tilde{\theta}^{**}\|_{\mathcal{X}} \le C(\|\tilde{\theta}^*\|_{\mathcal{X}}^2 + \|\theta_{-1,c}\|_{\mathbf{H}^m}^2 + \|\theta_{1,c}\|_{\mathbf{H}^m}^2).$$

Thus we deduce that for sufficiently small ball $\mathbf{B}_r(0) \subset \mathbf{B}_1(0)$,

$$\|\tilde{\theta} - \theta_c\|_{\mathcal{X}} \le C(\|\theta_{-1,c}\|_{\mathbf{H}^m}^2 + \|\theta_{1,c}\|_{\mathbf{H}^m}^2). \tag{3.18}$$

Note that $\tilde{\theta} - \theta_c := (\ldots, , 0, \theta_{-1,c}, 0, \theta_{1,c}, 0, \ldots)$. Hence by (3.18), we obtain

$$\|\theta_c\|_{\mathcal{X}} \le C(\|\theta_{-1,c}\|_{\mathbf{H}^m}^2 + \|\theta_{1,c}\|_{\mathbf{H}^m}^2).$$

and

$$\|\tilde{\theta}\|_{\mathcal{X}} \le C(\|\theta_{-1,c}\|_{\mathbf{H}^m}^2 + \|\theta_{1,c}\|_{\mathbf{H}^m}^2).$$

This completes the proof. □

Now we complete the proof of Theorem 1.2.

Proof of Theorem 1.2 To prove Theorem 1.2, the rest remains to analyze the equation (3.5). We restate the equation:

$$(\pm i\omega - \mathbf{F})\theta_{\pm 1,c} = \mathbf{P}_{\pm 1,c} N_{\pm 1}(\theta).$$

It follows from $\theta_{-1} = \overline{\theta_1}$ and $N(\theta) = \overline{N(\theta)}$ that the "$-$" equation is the complex conjugate of the "$+$" equation. By Lemma 3.5, we can denote $\theta = \mathbf{J}\tilde{\theta}$ by means of $\tilde{\theta} = \Phi(\theta_c) = \Phi(\overline{\theta_{1,c}}, \theta_{1,c})$. Our target is to find (ω, β) close to (ω_0, β_c) and a nontrivial solution $\theta_{1,c} = \theta_{1,c}(x)$ of

$$-i\omega\theta_{1,c} + \mathbf{F}\theta_{1,c} + \mathbf{P}_{1,c} N_1(\mathbf{J}\Phi(\overline{\theta_{1,c}}, \theta_{1,c})) = 0. \tag{3.19}$$

Due to $\theta_{1,c} \in \mathrm{C}\psi^+$ and $\mathbf{F}\psi^+ = \lambda_0^+(\beta)\psi^+$, we can write $\theta_{1,c} = \eta\psi^+$. Then by (3.19), we obtain the equation

$$-i\omega\eta\psi^+ + \lambda_0^+(\beta)\eta\psi^+ + \mathbf{P}_{1,c} N_1(\mathbf{J}\Phi(\overline{\eta\psi^+}, \eta\psi^+)) = 0 \;\; for\ some\ \eta \in \mathrm{C} \setminus \{0\}. \tag{3.20}$$

For simplicity, we introduce $p_{1,c}$ by $\mathbf{P}_{1,c}\theta = p_{1,c}(\theta)\psi^+$. Then equation (3.20) can be written as

$$-i\omega\eta + \lambda_0^+(\beta)\eta + f(\beta, \eta) = 0, \;\; for\ some\ \eta \in \mathrm{C}, \tag{3.21}$$

where the cubic coefficient $\mu \neq 0$ in

$$f(\beta, \eta) := p_{1,c}\left(N_1(\mathbf{J}\Phi(\overline{\eta\psi^+}, \eta\psi^+))\right). \tag{3.22}$$

Note that

$$|p_{1,c}(\theta)| \leq C\|\mathbf{P}_{1,c}\theta\|_{\mathbf{H}^m} \leq C\|\theta\|_{\mathbf{H}^m}. \tag{3.23}$$

So by (3.16), (3.22)-(3.23), $N(\theta) = R(\theta)\cdot\nabla\theta$ and $\|\nabla\Theta\|_{\mathbf{L}^\infty} < \infty$ (see Theorem 1.1), we derive

$$\begin{aligned}
|p_{1,c}\left(N_1(\mathbf{J}\Phi(\overline{\eta\psi^+},\eta\psi^+)\right)| &\leq C\|N_1(\mathbf{J}\Phi(\overline{\eta\psi^+},\eta\psi^+)\|_{\mathbf{H}^m} \\
&\leq C\|\Phi(\overline{\eta\psi^+},\eta\psi^+)\|_{\mathcal{X}}\|\nabla\Theta\|_{\mathbf{L}^\infty} \\
&\leq C(\|\overline{\theta_{1,c}}\|^2_{\mathbf{H}^m} + \|\theta_{1,c}\|^2_{\mathbf{H}^m}) \\
&\leq C\|\eta\psi^+\|^2_{\mathbf{H}^m} \leq C|\eta|^2,
\end{aligned}$$

where we use the notation

$$\tilde{\theta} = \Phi(\theta_c) = \Phi(\overline{\eta\psi^+},\eta\psi^+).$$

Inspired by the classical Hopf-Bifurcation result [22], we employ the implicit function theorem to find real value solutions (i.e. find $\gamma = \eta \in \mathbf{R}$) of equation (3.21). Hence, we define the complex-valued smooth function

$$\Upsilon(\gamma;\varrho,\beta) := \begin{cases} -i(\omega_0+\varrho) + \lambda_0^+(\beta_c+\epsilon) - \gamma^{-1}f(\beta_c+\epsilon,\gamma), & \gamma \neq 0, \\ -i(\omega_0+\varrho) + \lambda_0^+(\beta_c+\epsilon), & \gamma = 0. \end{cases}$$

It follows from $\lambda_0^+(\beta_c) = i\omega_0$ that $\Upsilon(0,0,0) = 0$. Moreover, by assumption (H2) the Jacobi Matrix

$$\mathbf{D}_{\rho,\epsilon}\Upsilon(\gamma;\varrho,\epsilon)|_{\gamma=\varrho=\epsilon=0} = \begin{pmatrix} 0 & \frac{d}{d\beta}Re\lambda_0^+(\beta)|_{\beta=\beta_c} \\ -1 & \frac{d}{d\beta}Im\lambda_0^+(\beta)|_{\beta=\beta_c} \end{pmatrix}$$

with respect to ρ, ϵ has

$$det\mathbf{D}_{\rho,\epsilon}\Upsilon(\gamma;\varrho,\epsilon)|_{\gamma=\varrho=\epsilon=0} = \frac{d}{d\beta}Re\lambda_0^+(\beta)|_{\beta=\beta_c} > 0.$$

Thus, for sufficiently small $\gamma > 0$, we find a function $\gamma \mapsto (\varrho(\gamma),\epsilon(\gamma))$ with $\varrho(0) = \epsilon(0) = 0$ such that

$$-i(\omega_0+\varrho(\gamma)) + \lambda_0^+(\beta_c+\epsilon(\gamma)) - \gamma^{-1}f(\beta_c+\epsilon(\gamma),\gamma) = 0.$$

Note the degree of the nonlinearity in f. Then it follows from differentiating this equation that $\epsilon^i \neq 0$ for some first i. Hence, the function $\gamma \mapsto \epsilon(\gamma)$ is locally invertible, and in the form $\epsilon \mapsto \gamma(\epsilon)$. Hence the following equation holds

$$-i(\omega_0+\varrho(\gamma(\epsilon)))\gamma(\epsilon) + \lambda_0^+(\beta_c+\epsilon)\gamma(\epsilon) - f(\beta_c+\epsilon,\gamma(\epsilon)) = 0,$$

for sufficiently small $\epsilon > 0$.

Therefore we obtain the desired solutions of (3.19) by setting $\omega = \omega_0 + \varrho(\gamma(\epsilon))$, $\beta = \beta_c + \epsilon$ and $\theta_{1,c} = \gamma(\epsilon)\psi^+_{\beta_c+\epsilon}(x)$. This result combining with Lemma 3.5 and Theorem 1.1 finishes the proof of Theorem 1.2.

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